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# On $n$-Widths in Sobolev Spaces and Applications to Elliptic Boundary Value Problems*

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## 1. INTRODUCTION

The notion of $n$-width, introduced by Kolmogorov [14], has received considerable attention in recent years (see [8], [12], [13], [15] and references therein) within approximation theory. This notion gives a measure of the approximability of a set $\mathcal{Q}$ in a normed linear space $\mathcal{X}$ by subspaces $\mathcal{M}$ of dimension $n$. Indeed, we define the $n$-width, $d_n(\mathcal{Q})$, of $\mathcal{Q}$ in $\mathcal{X}$ as

$$d_n(\mathcal{Q}) = \inf\{E(\mathcal{Q}, \mathcal{M}) : \mathcal{M} \subset \mathcal{X}, \dim \mathcal{M} = n\}$$

where

$$E(\mathcal{Q}, \mathcal{M}) = \sup_{f \in \mathcal{Q}} \inf_{g \in \mathcal{M}} \| f - g \|$$

and we define an optimal approximating subspace $\mathcal{M}$ in $\mathcal{X}$ as any subspace of dimension $\leqslant n$ satisfying $d_n(\mathcal{Q}) = E(\mathcal{Q}, \mathcal{M})$.

In this paper we will be concerned with $n$-widths in Hilbert space, especially Sobolev spaces and $\mathcal{L}_2$ spaces, and we will make applications of $n$-width results to functional equations and elliptic boundary value problems. Such applications arise naturally because of the fundamental numerical relationship which exists between the $n$-widths of classes satisfying quadratic inequalities and the eigenvalues of the associated functional equation or boundary value problem. Such relationships characterize the $n$-widths, roughly speaking, as the square roots of the reciprocals of the positive eigenvalues arranged in increasing order according to multiplicity. Characterizations of this type were obtained in [12] in the setting of $\mathcal{L}_2(\Omega)$, $\Omega$ a bounded region in Euclidean space, and in [13] asymptotic estimates of these $n$-widths were obtained and an application was made to the estimation of the asymptotic

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distribution of eigenvalues of certain functional equations and related elliptic boundary value problems.

Section 2 of this paper extends the results of [12] in several directions. The most significant generalization is that $n$-widths of classes contained in Sobolev spaces $\mathscr{H}_k(\Omega)$ are computed in $\mathscr{H}_j(\Omega)$ for $j < k$. This includes the characterization of the $\mathscr{L}_2(\Omega)$ $n$-widths as a special case. Moreover, through the use of a famous inequality of Sobolev, this information regarding convergence in $\mathscr{H}_j(\Omega)$ provides information regarding uniform convergence. A number of other generalizations are mentioned in Section 2 which extend the $n$-width results to a wider class of domains, in particular to compact manifolds and to quasi-bounded domains in Euclidean space, and to a wider class of boundary value problems than considered in [12], including combinations of forced and natural boundary conditions.

In Section 3 the exact asymptotic order of the $n$-widths in $\mathscr{L}_2(\Omega)$ is obtained for a class of domains $\Omega$ in $\mathbf{R}^m$ which include the strongly Lipschitz domains. The widths in $\mathscr{L}_2(\Omega)$ are shown to be weakly asymptotic to $n^{-k/m}$, i.e.,

$$(c/n)^{k/m} + o(n^{-k/m}) \leqslant d_n \leqslant (c_1/n)^{k/m} + o(n^{-k/m})$$

which extends Theorem 1.1 of [13] while at the same time eliminating the awkward hypothesis on the multiplicity of the eigenvalues. Section 3 is closed with the result, valid if $\partial\Omega$ is sufficiently smooth, that the $n$-widths in $\mathscr{H}_j(\Omega)$ are of order $O(n^{-(k-j)/m})$.

Section 4 completes the paper by constructing an optimal approximation scheme for the solution of elliptic boundary value problems for domains $\Omega$ with sufficiently smooth boundary, and, more generally, for the solution of of certain functional equations in Sobolev space.

## 2. $n$-Widths in Sobolev Spaces

Let $\Omega$ be a region in $\mathbf{R}^m$ or a compact $m$-dimensional $C^\infty$ Riemannian manifold with or without boundary and let $\mathscr{L}_2(\Omega)$ be the Hilbert space of complex-valued square integrable functions on $\Omega$ with inner product

$$(f, g) = \int_\Omega f\bar{g}\, d\Omega$$

(For a discussion of compact Riemannian manifolds, integration on such and associated function spaces see [7, Sec. 3.2], [16, Ch. 7] and [11, Secs. 1.8 and 2.6]) corresponding to the Riemannian structure. The reader may visualize, for example, open connected sets $\Omega$ in $\mathbf{R}^m$ with Lebesgue measure or spheres $S^m$ in $\mathbf{R}^{m+1}$ with spherical measure and the associated $\mathscr{L}_2$ spaces.

For $k \geqslant 0$ the Sobolev spaces $\mathscr{H}_k(\Omega)$ consist of those functions $f$ on $\Omega$ with distribution derivatives $D^\alpha f \in \mathscr{L}_2$ for all $0 \leqslant |\alpha| \leqslant k$ where

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m), \qquad |\alpha| = \sum_{i=1}^{m} \alpha_i .$$

Under the inner product

$$(f, g)_{\mathscr{H}_k} = \sum_{0 \leqslant |\alpha| \leqslant k} (D^\alpha f, D^\alpha g)_{\mathscr{L}_2} \tag{2.1.}$$

the spaces $\mathscr{H}_k$ are well-known to be Hilbert spaces. The space $\mathscr{H}_0(\Omega)$ is clearly the space $\mathscr{L}_2(\Omega)$.

It is well-known that the functions $C^\infty(\Omega)$ are dense in $\mathscr{H}_k(\Omega)$. The closure of the class $C_c^\infty(\Omega)$ of $C^\infty(\Omega)$ functions with compact support in $\Omega$ is designated by $\mathring{\mathscr{H}}_k(\Omega)$. In the special case that the boundary $\partial\Omega$ of $\Omega$ is empty, e.g., $\Omega = \mathbf{R}^m$ or $\Omega$ a compact manifold without boundary, then $\mathscr{H}_k(\Omega) = \mathring{\mathscr{H}}_k(\Omega)$. It is known that, if $\Omega$ is a bounded region in $\mathbf{R}^m$, the natural injections

$$I : \mathring{\mathscr{H}}_k(\Omega) \to \mathring{\mathscr{H}}_j(\Omega)$$

are compact for $j < k$. Moreover, if $\Omega$ is any compact manifold, with or without boundary or a bounded region in Euclidean space satisfying certain assumptions on the geometry of $\partial\Omega$, then the injections

$$I : \mathscr{H}_k(\Omega) \to \mathscr{H}_j(\Omega) \tag{2.2}$$

are compact for such regions $\Omega$ (see [11, Th. 2.6.3] where this result is proved for compact manifolds and [1, Th. 3.8] for the case of certain regions in $\mathbf{R}^m$, including those which satisfy the restricted cone condition.)

Let $B(u, v)$ be a positive, Hermitian, coercive form on a Hilbert space $\mathscr{V}_k$, $\mathring{\mathscr{H}}_k \subset \mathscr{V}_k \subset \mathscr{H}_k$, given by

$$B(u, v) = \sum_{\substack{0 \leqslant |\alpha| \leqslant k \\ 0 \leqslant |\beta| \leqslant k}} (D^\alpha u, a_{\alpha\beta} D^\beta v)_{\mathscr{L}_2} \tag{2.3}$$

where the $a_{\alpha\beta}$ are measureable coefficients and, as multipliers, are bounded operators on $\mathscr{L}_2$. Thus, $B(u, v)$ is assumed to satisfy, for all $u, v \in \mathscr{V}_k(\Omega)$,

$$B(u, u) \geqslant 0, \qquad B(u, v) = \overline{B(v, u)} \tag{2.4}$$

$$B(u, u) \geqslant c(u, u)_{\mathscr{H}_k} - \mu(u, u)_{\mathscr{L}_2} \tag{2.5}$$

where $c > 0$ and $\mu \geqslant 0$. The assumptions (2.4) and (2.5) clearly imply, combined with the assumptions on the $a_{\alpha\beta}$, that the norm

$$\{B(u, u) + \mu(u, u)_{\mathscr{L}_2}\}^{1/2} \tag{2.6}$$

on $\mathscr{V}_k$ is equivalent to the norm induced on $\mathscr{V}_k$ by (2.1).

The problem considered in this section is that of characterizing the $n$-widths in $\mathscr{H}_j(\Omega)$, for $0 \leqslant j < k$, of the classes

$$\mathscr{R}_k = \{f \in \mathscr{V}_k : B(f, f) \leqslant 1\}. \tag{2.7}$$

For convenience, we introduce the notation $\mathscr{V}_{k,j}$ to denote the closure of $\mathscr{V}_k$ in $\mathscr{H}_j$. Notice that $\mathscr{V}_{k,0} = \mathscr{L}_2$.

THEOREM 2.1. *There exists a positive self-adjoint operator $R$, from $\mathscr{V}_k$ to $\mathscr{H}_j$, satisfying*

$$B(u, v) = (Ru, v)_{\mathscr{H}}, \textit{for all } u \in \mathscr{D}_R,\ v \in \mathscr{V}_k.$$

*If the injection*

$$I : \mathscr{V}_k \to \mathscr{H}_j \tag{2.8}$$

*is compact then the equation*

$$B(\varphi, v) = \lambda(\varphi, v)_{\mathscr{H}}, \textit{for all } v \in \mathscr{V}_k \tag{2.9}$$

*has, for a sequence $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ of nonnegative values $\lambda_\nu$ tending to infinity, corresponding (one-dimensional) eigenmanifolds $\mathscr{M}_\nu$ which are orthogonal and dense in both $\mathscr{V}_{k,j}$ and $\mathscr{H}_k$. In this case the n-widths of $\mathscr{R}_k$ in the space $\mathscr{H}_j$, satisfy*

$$d_n(\mathscr{R}_k) = \infty \qquad \text{if} \qquad n < N \tag{2.10}$$

$$d_n(\mathscr{R}_k) = \lambda_{n+1}^{-1/2} \qquad \text{if} \qquad n \geqslant N \tag{2.11}$$

$$d_n(\mathscr{R}_k) = E(\mathscr{R}_k, \mathscr{M}_1 + \cdots + \mathscr{M}_n) \tag{2.12}$$

*where $N$ satisfies $0 = \lambda_N < \lambda_{N+1}$ if $N > 0$.*

PROOF. The existence of $R$ and the properties of the eigenfunctions of (2.9) follow from Theorem 3.2 of [12] upon identifying $\mathscr{H}_0$ with $\mathscr{V}_k$ and $\mathscr{H}$ with $\mathscr{V}_{k,j}$. (2.10) through (2.12) are valid expressions in the space $\mathscr{V}_{k,j}$ for the numbers $d_n(\mathscr{R}_k)$, as follows directly from Theorem 3.3 of [12]. To show that (2.10) through (2.12) are valid assertions in the space $\mathscr{H}_j(\Omega)$, as the theorem states, it suffices to show that, for each $n > 0$, there does not exist an $n$-dimensional space $\mathscr{M}(n)$ in $\mathscr{H}_j$ satisfying

$$E(\mathscr{R}_k, \mathscr{M}(n)) < \begin{cases} \infty & \text{if} \quad 0 < n < N \\ \lambda_{n+1}^{-1/2} & \text{if} \quad n \geqslant N. \end{cases}$$

Now, the case $n < N$ is immediate. Indeed, $E(\mathscr{R}_k, \mathscr{M}(n)) = \infty$ for every subspace $\mathscr{M}(n)$ of $\mathscr{H}_j$ of dimension $n$ since $\mathscr{R}_k$ contains the $N$-dimensional subspace, $\mathscr{M}_1 + \cdots + \mathscr{M}_N$. Suppose then that $n \geqslant N$ and let $\mathscr{M}(n)$ be any $n$-dimensional subspace of $\mathscr{H}_j$. We will show that

$$E(\mathscr{R}_k, \mathscr{M}(n)) \geqslant E(\mathscr{R}_k, \mathscr{M}_1 + \cdots + \mathscr{M}_n).$$

Now the projection of the subspace $\mathscr{M}(n)$ onto $\mathscr{V}_{k,j}$ is a subspace of dimension $\nu \leqslant n$. Denoting by $g(f)$ the element in $\mathscr{M}(n)$ of best approximation to $f \in \mathscr{R}_k$ and by $v(f)$ the projection of $g(f)$ onto $\mathscr{V}_{k,j}$ we have the following inequalities:

$$\begin{aligned} E(\mathscr{R}_k, \mathscr{M}(n)) = \sup_{f \in \mathscr{R}_k} \|f - g(f)\| \geqslant \sup_{f \in \mathscr{R}_k} \|f - v(f)\| &\geqslant E(\mathscr{R}_k, \mathscr{M}_1 + \cdots + \mathscr{M}_\nu) \\ &\geqslant E(\mathscr{R}_k, \mathscr{M}_1 + \cdots + \mathscr{M}_n). \end{aligned}$$

This concludes the proof of the theorem.

We will now draw a number of corollaries from Theorem 2.1.

For bounded regions which satisfy the restricted cone property, Sobolev's embedding theorem and inequality are valid ([1, Theorem 3.9]). If $k > [m/2]$ and $l_k = k - [m/2] - 1$ then

$$\mathscr{H}_k(\Omega) \subset C^{l_k}(\bar{\Omega}).$$

In particular, it follows from this embedding result that $\mathscr{R}_k$ and the eigenfunctions of (2.9) are $C^{l_k}(\bar{\Omega})$ functions. Moreover, if $j > [m/2]$, then there exists a constant $\gamma$ depending only on $\Omega$ and $j$ such that for every $u \in \mathscr{H}_j(\Omega)$,

$$\| u \|_{C^{l_j}} \leqslant \gamma \| u \|_{\mathscr{H}},$$

where

$$\| u \|_{C^{l_j}} = \sum_{0 \leqslant |\alpha| \leqslant l_j} \sup_{x \in \Omega} | D^\alpha u(x)|. \tag{2.13}$$

From this discussion and from Theorem 2.1 we have

COROLLARY 2.2. *Suppose $\Omega$ is a bounded region in $\mathbf{R}^m$ satisfying the restricted cone condition and let $k > [m/2]$ with $l_k = k - [m/2] - 1$. Then the n-widths of $\mathscr{R}_k$ in $\mathscr{H}_j(\Omega)$ for $j < k$ satisfy* (2.10) *through* (2.12) *and moreover the optimal approximating manifold $\mathscr{M}_1 + \cdots + \mathscr{M}_n$ consists of $C^{l_k}(\bar{\Omega})$ functions. If $j > [m/2]$ then the n-widths of $\mathscr{R}_k$ in the Banach space $C^{l_j}(\bar{\Omega})$, with norm given by* (2.13), *are of order*

$$O(\lambda_{n+1}^{-1/2})$$

*where the $\lambda_n$ satisfy* (2.9).

If $j = 0$ and the coefficients $a_{\alpha\beta}$ are sufficiently regular then the operator $R$ of Theorem 2.1 is easily seen to have the special form

$$R\psi = \sum_{\substack{0 \leqslant |\alpha| \leqslant k \\ 0 \leqslant |\beta| \leqslant k}} D^{\alpha}[a_{\alpha\beta} D^{\beta}\psi]$$

for all $\psi \in C_c^{\infty}(\Omega)$. From this fact and the theory of distributions we have (see [11, Sec. 7.4] and, for real $a_{\alpha\beta}$, [19, p. 178])

COROLLARY 2.3. *If the coefficients $a_{\alpha\beta}$ of $B(u, v)$ are in $C^k(\bar{\Omega}) \cap C^{\infty}(\Omega)$ and the natural injection* (2.8) *is compact for $j = 0$ then the eigenfunctions $\varphi_\nu$ of* (2.9) *for $j = 0$ are $C^{\infty}(\Omega)$ functions and satisfy the uniformly elliptic partial differential equation in $\Omega$:*

$$\sum_{\substack{0 \leqslant |\alpha| \leqslant k \\ 0 \leqslant |\beta| \leqslant k}} D^{\alpha}[a_{\alpha\beta} D^{\beta}_{\varphi_\nu}] = \lambda_\nu \varphi_\nu . \tag{2.14}$$

From the preceding discussion and from Theorem 2.1 and Corollary 2.3 we have

COROLLARY 2.4. *If $\Omega$ is a compact $C^{\infty}$ manifold or a compact $C^{\infty}$ manifold with boundary then the injections* (2.2) *are compact and* (2.10) *through* (2.12) *hold. Moreover, if $j = 0$, $a_{\alpha\beta} \in C^{\infty}(\Omega)$ and $\partial\Omega$ is empty then* (2.14) *is equivalent to* (2.9).

As an illustration of Corollary 2.4 we consider a case where $\Omega$ is a compact $C^{\infty}$ manifold without boundary. In particular, if $\Omega =$ unit sphere in $\mathbf{R}^{m+1}$ and

$$B(u, v) = \int_{\Omega} \left( \sum_{|\alpha| = k} \frac{|\,a\,|!}{\alpha_1! \cdots \alpha_m!} D^{\alpha}u \cdot D^{\alpha}\bar{v} \right) d\Omega$$

then $B(u, v)$ is a positive coercive form satisfying (2.4) and (2.5) with $\mu = 0$ and (2.7) is equivalent to the polyharmonic eigenvalue problem on $\Omega$,

$$(-1)^k \left( \sum_{j=1}^{m+1} \frac{\partial^2}{\partial x_j^2} \right)^k \varphi_\nu = \lambda_\nu \varphi_\nu .$$

It follows that the $\varphi_\nu$ are the generalized (to $k \geqslant 1$) ultraspherical polynomials and the $\lambda_\nu$ are simply $\{j^k(j + m - 1)^k\}_{j=0}^{\infty}$ written in increasing order with multiplicity $N(j) - N(j - 2k)$ (for $j - m > k$) where

$$N(i) = 0 \qquad i < 0$$
$$N(i) = \frac{(i + m)!}{i!\, m!} \qquad i \geqslant 0.$$

(See [17] for a discussion of the properties of polyharmonic homogeneous polynomials.)

A region $\Omega$ in $\mathbf{R}^m$ is said to be quasi-bounded if there exist numbers $d(r)$ and $\delta(r)$, defined for $r \geqslant 0$, satisfying

(a) for each $x \in \Omega$ with $|x| > r$ there is a point $y$ such that $|x - y| < d(r)$ and

$$\Omega \cap \{z : |z - y| < \delta(r)\} = \varnothing; \tag{2.15}$$

(b) $$d(r)/\delta(r) \leqslant M < \infty \qquad \text{for all} \qquad r \geqslant 0;$$

(c) $$d(r) \to 0 \qquad \text{as} \qquad r \to \infty.$$

Clark [6] has shown that if $\Omega$ is a quasi-bounded region in $\mathbf{R}^m$ then the embeddings

$$I : \mathring{\mathscr{H}}_k(\Omega) \to \mathring{\mathscr{H}}_j(\Omega)$$

are compact for $j < k$. Thus we have

COROLLARY 2.5. *If $\Omega$ is a quasi-bounded region then* (2.10) *through* (2.12) *hold provided $\mathscr{V}_k = \mathring{\mathscr{H}}_k$.*

We wish to describe now situations in which the functional equation (2.9) with $j = 0$ is equivalent to an elliptic eigenvalue problem. The material is taken essentially from [1]. Suppose $\Omega$ is a bounded region in $\mathbf{R}^m$ with $\partial\Omega$ sufficiently smooth so that linear differential boundary operators $B_0, B_1, \ldots, B_{\nu-1}$ may be defined on $\partial\Omega$ where $\nu \leqslant k$ and the order of $B_i$ is less than $k$. We assume that the orders of $B_0, \ldots, B_{\nu-1}$ are distinct and that $\partial\Omega$ is non-characteristic for these operators. Let $\mathscr{V}_k$ be the closure in $\mathscr{H}_k(\Omega)$ of those $C^k(\bar{\Omega})$ functions $\varphi$ satisfying $B_i\varphi = 0$ on $\partial\Omega$ for $0 \leqslant i \leqslant \nu - 1$. Notice that $\mathring{\mathscr{H}}_k(\Omega) \subset \mathscr{V}_k(\Omega) \subset \mathscr{H}_k(\Omega)$. Indeed, additional boundary operators $B_\nu, B_{\nu+1}, \ldots, B_{k-1}$ on $\partial\Omega$ can be determined provided $\partial\Omega$ is sufficiently smooth such that $\partial\Omega$ is not characteristic for these operators and, if $0 \leqslant j \leqslant k - 1$, then, for some $i$, $B_i$ has order $j$. Elementary arguments then show that the completion of those $C^k(\bar{\Omega})$ functions satisfying $B_i\varphi = 0$ on $\partial\Omega$ for $0 \leqslant i \leqslant k - 1$ is exactly $\mathring{\mathscr{H}}_k(\Omega)$ and, of course, the completion of $C^k(\bar{\Omega})$ is just $\mathscr{H}_k(\Omega)$.

If we index the $B_i$, $0 \leqslant i \leqslant k - 1$, according to order it follows that there exist natural boundary operators $M_{2k-i-1}$ of order $2k - i - 1$, $0 \leqslant i \leqslant k - 1$, for which $\partial\Omega$ is not characteristic and for which the integration by parts formula

$$B(v, u) = (v, Au)_{\mathscr{L}_2} + \sum_{j=0}^{k-1} \int_{\partial\Omega} B_j v \, \overline{M_{2k-j-1} u} \, d\sigma$$

is valid for $u, v \in \mathscr{V}_k$ where

$$Au = \sum_{\substack{0 \leqslant |\alpha| \leqslant k \\ 0 \leqslant |\beta| \leqslant k}} D^\alpha [a_{\alpha\beta} D^\beta u].$$

$A$ agrees with the operator $R$ of Theorem 2.1 on $\mathscr{V}_k$ and hence

$$\sum_{j=0}^{k-1} \int_{\partial\Omega} B_j v \, \overline{M_{2k-j-1} u d\sigma} = 0 \qquad \text{for} \qquad v, u \in \mathscr{V}_k . \tag{2.16}$$

If we denote by $\Lambda$ the set of indices $i$ for which $B_i$ is originally specified then it follows from (2.16) that $M_{2k-i-1} u = 0$ on $\partial\Omega$ if $i \notin \Lambda$. $\mathscr{V}_k$ is then seen to be the completion of those $C^k(\bar{\Omega})$ functions such that $B_i u = 0$ on $\partial\Omega$ if $i \in \Lambda$ and $M_{2k-i-1} u = 0$ on $\partial\Omega$ if $i \notin \Lambda$. Hence we have, for a bounded region $\Omega$ in $\mathbf{R}^m$, where we may take $\Omega \in C^{2k}$ and $a_{\alpha\beta} \in C^k(\bar{\Omega})$,

COROLLARY 2.6. *If $\partial\Omega$ and $a_{\alpha\beta}$ are sufficiently regular then the n-widths of $\mathscr{R}_k$ in $\mathscr{L}_2(\Omega)$ satisfy* (2.10) *through* (2.12) *where the eigenvalues involved are those of the elliptic boundary value problem*

$$\begin{array}{llllll} \text{(a)} & Av = \lambda v & \textit{in} & \Omega & & \\ \text{(b)} & B_i v = 0 & \textit{on} & \partial\Omega & \textit{for} & i \in \Lambda \\ \text{(c)} & M_{2k-i-1} v = 0 & \textit{on} & \partial\Omega & \textit{for} & i \notin \Lambda. \end{array} \tag{2.17}$$

In [2], Aubin has characterized the $n$-widths of classes similar to those determined by the coercive forms $B(u, u)$ for a class of interpolation Hilbert spaces. His characterization theorems are also in terms of the eigenvalues of positive self-adjoint operators.

## 3. ASYMPTOTIC ESTIMATES

In this section we will sharpen the asymptotic theorems on $n$-widths and eigenvalues as contained in [13]. The first result is contained in

THEOREM 3.1. *Let $\Omega$ be a region in $\mathbf{R}^m$, suppose that the injection* (2.8) *is compact and let $B(u, v)$ be a positive, Hermitian, coercive form on $\mathscr{V}_k$, i.e., satisfying* (2.4) *and* (2.5). *Suppose there exists a domain $\Omega' \supset \Omega$ such that the injection $I : \mathring{\mathscr{H}}_k(\Omega') \to \mathscr{L}_2(\Omega')$ is compact and such that there exists a bounded linear extension operator $E$,*

$$E : \mathscr{V}_k(\Omega) \to \mathring{\mathscr{H}}_k(\Omega'),$$

*satisfying* $Ef(x) = f(x)$ a.e. *in* $\Omega$. *Then if* $B(u, v)$ *satisfies* (2.4) *and* (2.5) *over* $\mathring{\mathscr{H}}_k(\Omega')$ *and if the eigenvalues of the functional equations*

$$\begin{aligned} &\text{(a)}\ B(\varphi, v) = \lambda(\varphi, v)_{\mathscr{L}_2(\Omega)} \qquad \textit{for all} \qquad v \in \mathring{\mathscr{H}}_k(\Omega), \\ &\text{(b)}\ B(\varphi, v) = \lambda(\varphi, v)_{\mathscr{L}_2(\Omega')} \qquad \textit{for all} \qquad v \in \mathring{\mathscr{H}}_k(\Omega'), \end{aligned} \tag{3.1}$$

*have the asymptotic distributions*

$$\begin{aligned} &\text{(a)}\ N(\lambda) = c\lambda^{m/2k} + o(\lambda^{m/2k}) \qquad \lambda \to \infty, \\ &\text{(b)}\ N(\lambda) = c'\lambda^{m/2k} + o(\lambda^{m/2k}) \qquad \lambda \to \infty, \end{aligned} \tag{3.2}$$

*respectively for the number* $N(\lambda)$ *of eigenvalues* $\leqslant \lambda$ *counted according to multiplicity then the n-widths* $d_n(\mathscr{R}_k(\Omega))$ *have the following asymptotic estimates*:

$$d_n(\mathscr{R}_k) \geqslant (c/n)^{k/m} + o(n^{-k/m}) \qquad n \to \infty \tag{3.3}$$

$$d_n(\mathscr{R}_k) \leqslant (c_1/n)^{k/m} + o(n^{-k/m}) \qquad n \to \infty \tag{3.4}$$

for some $c_1 \geqslant c$. Equivalently,

$$c \leqslant \varliminf_{n\to\infty} d_n(\mathscr{R}_k)\, n^{k/m} \leqslant \varlimsup_{n\to\infty} d_n(\mathscr{R}_k)\, n^{k/m} \leqslant c_1 . \tag{3.5}$$

For the proof of this theorem we need a lemma, privately communicated to the author by Professor Colin Clark, and a second lemma which can be viewed as a strengthening of Lemma 2.2 of [13].

LEMMA 3.2. *If the eigenvalues* $0 \leqslant \mu_1 < \mu_2 < \cdots$ *of* (3.1a) *or* (3.1b) *are enumerated distinctly and are of multiplicity* $\gamma_i$ *and if* $N_n = \sum_{i=1}^n \gamma_i$ *then* $N_{n+1} = N_n(1 + o(1))$ *if* $N(\lambda)$ *satisfies* (3.2a) *or* (3.2b), *respectively.*

*Proof of Lemma* 3.2. Set $\alpha = m/2k$. For each $0 < \epsilon < 1$ there exists $\lambda_\epsilon$ such that

$$(1 - \epsilon)\, c\lambda^\alpha \leqslant N(\lambda) \leqslant (1 + \epsilon)\, c\lambda^\alpha \qquad \lambda \geqslant \lambda_\epsilon . \tag{3.6}$$

Define

$$\gamma(\lambda) = N(\lambda+) - N(\lambda-).$$

Then,

$$\gamma(\lambda) \leqslant 2\epsilon c\lambda^\alpha \qquad \lambda \geqslant \lambda_\epsilon . \tag{3.7}$$

By (3.6) we have

$$\lambda^\alpha \leqslant \frac{N(\lambda^-)}{(1 - \epsilon)\, c} \qquad \lambda \geqslant \lambda_\epsilon . \tag{3.8}$$

Setting $\lambda = \mu_n$ in (3.7) and 3.8) we have for large $n$,

$$\gamma_n = \gamma(\mu_n) \leqslant \frac{2\epsilon}{(1-\epsilon)} N(\mu_n^-) = \frac{2\epsilon}{(1-\epsilon)} (\gamma_1 + \cdots + \gamma_{n-1}).$$

Hence,

$$1 \leqslant \frac{\gamma_1 + \cdots + \gamma_n}{\gamma_1 + \cdots + \gamma_{n-1}} \leqslant 1 + \frac{2\epsilon}{(1-\epsilon)}, \qquad \text{i.e.,} \qquad N_n = N_{n-1}(1 + o(1)).$$

This completes the proof of Lemma 3.2.

LEMMA 3.3. *Under the hypotheses of Theorem* 3.1, *except for the assumption of the existence of the operator E, we have the following relations*

$$d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega)) = (c/n)^{k/m} + o(n^{-k/m}) \qquad as \qquad n \to \infty$$

$$d_n(\mathring{\mathscr{R}}_k) = (c'/n)^{k/m} + o(n^{-k/m}) \qquad as \qquad n \to \infty$$

*where* $\mathring{\mathscr{R}}_k = \{f \in \mathring{\mathscr{H}}_k(\Omega') : B(f, f) \leqslant 1\}$.

*Proof.* Since the proofs of the two relations are identical, we will prove only the first. Now, if $N_\nu \leqslant n < N_{\nu+1}$, where the $N_\nu$ are defined in Lemma 3.2, we have from (2.11), Lemma 3.2 and (3.2b),

$$\begin{aligned} \underline{\lim}\, d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega))\, n^{k/m} &\geqslant \underline{\lim}\, \mu_{\nu+1}^{-1/2} N_\nu^{k/m} \\ &= \underline{\lim} \left(\frac{N_\nu}{N_{\nu+1}}\right)^{k/m} N_{\nu+1}^{k/m} \mu_{\nu+1}^{-1/2} \\ &= \underline{\lim}\, N_{\nu+1}^{k/m} \mu_{\nu+1}^{-1/2} \\ &= c. \end{aligned}$$

Similarly,

$$\begin{aligned} \overline{\lim}\, d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega))\, n^{k/m} &\leqslant \overline{\lim}\, \mu_{\nu+1}^{-1/2} N_{\nu+1}^{k/m} \\ &= c. \end{aligned}$$

It follows that $c = \lim d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega))\, n^{k/m}$ and the lemma is proved.

*Proof of Theorem* 3.1. $d_n(\mathscr{R}_k) \geqslant d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega))$ since $n$-widths increase with the size of the function class. But since the eigenvalues of (3.1a) over $\mathring{\mathscr{H}}_k(\Omega)$ satisfy (3.2a) it follows from Lemma 3.3 that

$$d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega)) = (c/n)^{k/m} + o(n^{-k/m})$$

and (3.3) follows.

To obtain (3.4) we notice that the $n$-widths of the class

$$\mathring{\mathscr{R}}_k = \{f \in \mathring{\mathscr{H}}_k(\Omega') : B(f, f) \leqslant 1\}$$

satisfy (2.10) through (2.12) since $I : \mathring{\mathscr{H}}_k(\Omega') \to \mathscr{L}_2(\Omega')$ is assumed compact. Since the eigenvalue distribution $N(\lambda)$ for the eigenvalues of (3.1b) over $\mathring{\mathscr{H}}_k(\Omega')$ is assumed to satisfy (3.2b) for some constant $c'$ it follows from Lemma 3.3

$$d_n(\mathring{\mathscr{R}}_k) = (c'/n)^{k/m} + o(n^{-k/m}) \qquad as \qquad n \to \infty.$$

Finally, by the assumption of the existence of the extension operator $E$ it follows from Lemma 2.6 of [13] that $d_n(\mathscr{R}_k) \leqslant \kappa d_n(\mathring{\mathscr{R}}_k)$ and (3.4) follows with $c_1 = \kappa^{m/k} c'$.

(3.5) is simply a restatement of (3.3) and (3.4).

COROLLARY 3.4. *If $\Omega$ is a bounded strongly Lipschitz domain and $a_{\alpha\beta} \in C^\infty(\bar{\Omega})$, and if $B(u, v)$ satisfies the hypotheses* (2.4) *and* (2.5), *then the asymptotic estimates of $d_n(\mathscr{R}_k)$ satisfy* (3.5), *provided* (2.4) *and* (2.5) *hold in some $\Omega' \supset \bar{\Omega}$.*

*Proof.* The proof of Corollary 3.4 will follow the lines of the proof of Theorem 3.1. Indeed, the assumption that $\Omega$ is a bounded, uniformly Lipschitz (or regular) region implies that, given any bounded region $\Omega' \supset \bar{\Omega}$, there exists a bounded extension operator $E : \mathscr{H}_k(\Omega) \to \mathring{\mathscr{H}}_k(\Omega')$ (see [3, Theorem 3.4.3, p. 74]). We will now show that the $d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega))$ have the asymptotic form $(c/n)^{k/m} + o(n^{-k/m})$ as $n \to \infty$ with a similar relation for $d_n(\mathring{\mathscr{R}}_k)$ in the space $\mathring{\mathscr{H}}_k(\Omega')$ for a suitably chosen region $\Omega'$. These two relations, together with the extension operator $E$, were precisely the tools used to prove Theorem 3.1.

Now by Lemma 2.1 of [13], it follows that the eigenvalues of (3.1a) have the asymptotic distribution of (3.2a) in the special case $\Omega \in C^{2ks}$, where $s > 1 + (m + 1)/2k$ and where the constant $c$ is defined by

$$c = (2\pi)^{-m} \int_\Omega w(x)$$

$$w(x) = \text{measure}\left\{\xi : 0 < \sum_{\substack{|\alpha|=k \\ |\beta|=k}} a_{\alpha\beta}(x)\, \xi^{\alpha+\beta} < 1\right\}.$$

Hence, since the injection $I : \mathring{\mathscr{H}}_k(\Omega) \to \mathscr{L}_2(\Omega)$ is compact, it follows from Lemma 3.3 that the $d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega))$ have the appropriate form in this case. Now, a standard, though technical, argument shows that a uniformly Lipschitz domain $\Omega$ can be approximated arbitrarily closely from within and without by $C^{2ks}$ domains, i.e., given $\epsilon > 0$, there exist $C^{2ks}$ domains $\Omega_i \subset \Omega \subset \Omega_0$ such that measure $(\Omega - \Omega_i) < \epsilon$ and measure $(\Omega_0 - \Omega_i) < \epsilon$.

It follows that

$$c_i \leqslant \liminf_{n\to\infty} d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega))\, n^{k/m} \leqslant \limsup_{n\to\infty} d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega)\, n^{k/m} \leqslant c_0$$

where

$$c_i = \int_{\Omega_i} w(x) \leqslant \int_{\Omega} w(x) \leqslant \int_{\Omega_0} w(x) = c_0 .$$

Since $c_i(\epsilon)\nearrow$, $c_0(\epsilon)\searrow$ and $c_0(\epsilon) - c_i(\epsilon) \to 0$ as $\epsilon \to 0$ it follows that

$$\lim_{n\to\infty} d_n(\mathscr{R}_k \cap \mathring{\mathscr{H}}_k(\Omega))\, n^{k/m} = c$$

where $c = \int_\Omega w(x)$. Since the results for $d_n(\mathring{\mathscr{R}}_k)$ are immediate for any regular $\Omega' \supset \bar{\Omega}$, the proof of Corollary 3.4 is concluded.

Corollary 3.4 is actually a strengthened version of Theorem 1.1 of [13]. Some statements can be made for quasi-bounded regions $\Omega$ in the special case that

$$B(u, v) = \sum_{i=1}^{m} \int_\Omega \left(\frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i}\right) d\Omega \tag{3.9}$$

and $\mathscr{V}_k = \mathring{\mathscr{H}}_k(\Omega)$ due to results of Clark [5] on the distribution of the eigenvalues of the Laplacian operator in this case. Indeed, from Lemma 3.3 we have

COROLLARY 3.5. *Suppose $\Omega$ is a quasi-bounded region and $B(u, v)$ satisfies* (3.9). *Then, if $\mathscr{V}_k = \mathring{\mathscr{H}}_1(\Omega)$ and $\Omega$ has finite measure, the n-widths of $\mathscr{R}_1$ satisfy*

$$d_n(\mathscr{R}_1) = (c/n)^{-1/m} + o(n^{-1/m}) \qquad as \qquad n \to \infty \tag{3.10}$$

We will close Section 3 by applying the asymptotic estimates for $\mathscr{L}_2$ $n$-widths to obtain asymptotic estimates for $\mathscr{H}_j$ $n$-widths. The chief tool used is the theory of interpolation spaces which is developed in [10] in the setting of Sobolev spaces.

Specifically, let $\Omega$ be a bounded region with sufficiently smooth boundary and let $\mathscr{V}_k$ be the completion in $\mathscr{H}_k(\Omega)$ of those $C^k(\bar{\Omega})$ functions $u$ satisfying $B_i u = 0$ on $\partial\Omega$ if $i \in \Lambda$ and $M_{2k-i-1}u = 0$ if $i \notin \Lambda$ as in section 2. Then $\mathscr{V}_k$ is contained in the subspace $\mathscr{V}_j$ of $\mathscr{H}_j(\Omega)$ consisting of the completion in $\mathscr{H}_j(\Omega)$ of those $u$ in $C^j(\bar{\Omega})$ satisfying $B_i u = 0$ on $\partial\Omega$ if $i < j$ and $i \in \Lambda$ and $M_{2j-i-1}u = 0$ on $\partial\Omega$ if $i < j$ and $i \notin \Lambda$. According to Theorem 8.1 of [10], $\mathscr{V}_j$ is a Hilbert space interpolating $\mathscr{V}_k$ and $\mathscr{L}_2$ with exponent $1 - j/k$. Moreover, for each $u \in \mathscr{V}_k$

$$\| u \|_{\mathscr{H}_j} \leqslant (\| u \|_{\mathscr{H}_k})^{j/k} (\| u \|_{\mathscr{L}_2})^{1-j/k}. \tag{3.11}$$

An immediate consequence of this discussion and Corollary 3.4 is the

THEOREM 3.6. *Let $\Omega$ be a bounded region with sufficiently smooth boundary e.g., $\Omega \in C^{2k}$. Then the n-widths, $d_n(\mathscr{R}_k)$, in $\mathscr{H}_j(\Omega)$ satisfy*

$$d_n(\mathscr{R}_k) = O(n^{-(k-j)/m}) \qquad \text{as} \qquad n \to \infty. \tag{3.12}$$

## 4. APPROXIMATION SCHEMES

In this last section we will outline an approximation scheme for the solutions of functional equtions of the form

$$B(u, v) = (f, v)_{\mathscr{H}_j} \qquad \text{for all} \qquad v \in \mathscr{V}_k \tag{4.1}$$

where $\mathring{\mathscr{H}}_k(\Omega) \subset \mathscr{V}_k \subset \mathscr{H}_k(\Omega)$, $B(u, v)$ is a positive, Hermitian, coercive form on $\mathscr{V}_k$ and $f$ is in the closure $\mathscr{V}_{k,j}$ of $\mathscr{V}_k$ in $\mathscr{H}_j$. If $\partial\Omega$ is sufficiently smooth and if $j = 0$ in (4.1) this may be viewed as an approximation scheme for solutions of elliptic boundary value problems of the form

$$\begin{array}{llllll}
\text{(a)} & Au = f & \text{in} & \Omega & & \\
\text{(b)} & B_i u = 0 & \text{on} & \partial\Omega & \text{for} & i \in \Lambda \\
\text{(c)} & M_{2k-i-1} u = 0 & \text{on} & \partial\Omega & \text{for} & i \notin \Lambda
\end{array}$$

where $A$ is a uniformly strongly elliptic operator of order $2k$ determined by $B(u, v)$ as in (2.17), the $B_i$ and the $M_{2k-i-1}$ are boundary operators as found there, and $f$ is orthogonal in $\mathscr{L}_2$ to the null solutions $\varphi$ of $B(v, v)$.

If the injection (2.8) is compact then the approximation scheme may be defined in terms of the eigenvalues and eigenfunctions of (2.9) and the error may be estimated in terms of the norm of the inverse $G$ of the restriction to $\mathscr{D}_R \cap N_R^{\perp}$ of the operator $R$ of Theorem 2.1. Indeed, if $N$ satisfies $0 = \lambda_N < \lambda_{N+1}$ where $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ are the eigenvalues of (2.9) with corresponding eigenfunctions $\varphi_1, \varphi_2, \ldots$ then we have

THEOREM 4.1. *The sequence*

$$\psi_n = \sum_{i=N+1}^{n} \frac{1}{\lambda_i} (f, \varphi_i)_{\mathscr{H}_j} \varphi_i \tag{4.2}$$

*converges in $\mathscr{H}_j$ to a solution $u$ of* (4.1) *provided $f$ is orthogonal to $\{\varphi_1, \ldots, \varphi_N\}$ in $\mathscr{V}_{k,j}$. In this case every solution $\tilde{u}$ of* (4.1) *differs from $u$ by a linear combination of $\varphi_1, \ldots, \varphi_N$. The error, $\| u - \psi_n \|_{\mathscr{H}_j}$, may be estimated:*

$$\| u - \psi_n \|_{\mathscr{H}_j} \leqslant \sqrt{\| G \|}\, \| f \|_{\mathscr{H}_j} \lambda_{n+1}^{-1/2} \tag{4.3}$$

*where $G$ is the bounded operator from $\mathscr{V}_{k,j}$ to $\mathscr{D}_R \cap \{\varphi_1, \ldots, \varphi_N\}^\perp$, satisfying $RG = I$ on $\mathscr{V}_{k,j}$ and $GR = I$ on $\mathscr{D}_R \cap \{\varphi_1, \ldots, \varphi_N\}^\perp$, with the latter subspace of $\mathscr{H}_k$ normed by $\sqrt{B(v, v)}$.*

*Proof.* The sequence defined by (4.2) converges in $\mathscr{H}_k$ by the definition of $R$ and hence, a fortiori, in $\mathscr{H}_j$. The sequence $R\psi_n$ is simply the sequence of partial sums in the expansion of $f$ along $\{\varphi_n\}$ in $\mathscr{H}_j$. Clearly, $u$ is unique if and only if $R$ has no zero eigenvalues. (4.3) is simply a statement concerning the $n$-widths of the class

$$\mathscr{S} = \{v : B(v, v) \leqslant \| G \| \, \| f \|_{\mathscr{H}_j}^2\} \tag{4.4}$$

for the operator

$$G : \mathscr{H}_j \to \mathscr{D}_R \cap \{\varphi_1, \ldots, \varphi_N\}^\perp$$

satisfying $GR = I$ on $\mathscr{D}_R \cap \{\varphi_1, \ldots, \varphi_N\}^\perp$ and $RG = I$. This completes the proof of the theorem.

We remark that the approximations defined by (4.2) may be termed optimal in the sense that, for any subspace $\mathscr{M}$ of dimension $n$ in $\mathscr{V}_{k,j}$ such that the dispersion $E(\mathscr{S}, \mathscr{M})$ is minimal, the error bounds of (4.3) are sharp for $u \in \mathscr{S}$ and $\psi_n \in \mathscr{M}$.

Theorems 3.6 and 4.1 combine to yield the

COROLLARY 4.2. *The sequence defined by* (4.2) *converges in $\mathscr{H}_j$ with order $O(n^{-(k-j)/m})$ for $0 \leqslant j \leqslant k$, if $\Omega \in C^{2k}$.*

The results of Theorem 4.1, in the case $j = 0$, are similar to those of [3, Sec. 4.6] and [18] where optimal approximation of solutions of two point boundary value problems is considered. However, their treatment does not yield asymptotic error bounds as in Corollary 4.2 above and, of course, is specialized to the case $m = 1$ and $j = 0$.

It should be mentioned, finally, that the assumptions on $B(u, v)$ in Corollary 4.2 are exactly those of Corollary 3.4, and the smoothness assumptions on $\Omega$ may be considerably relaxed.

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